AN APPROXIMATE METHOD FOR COMPUTING ERROR COEFFICIENT MATRICES FOR LUNAR TRAJECTORIES

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Introduction

Studies conducted at Ames[1,2] on midcourse navigation for a circumlunar mission used linear matrix operations for trajectory determination and guidance. These operations require the knowledge of "error coefficient" or "transition" matrices, which are the matrices of first partial derivatives of the variational parameters at one time on a given reference trajectory with respect to those at an earlier time. The variational parameters may be any set of variables which completely defines deviations from the reference position and velocity.

This paper is concerned with the problems involved in the computation of the transition matrices. The results presented are from a theoretical study conducted in an attempt to produce a simpler, though perhaps less accurate, method of computation than that used previously. This alternate method is based on the patched conic approach. Although the results are inconclusive in many respects, it appears that the method can be used to advantage in some cases.

The work can be divided into four parts as follows: First, it is shown that computation of the matrices in closed form using patched conic approximations is inherently limited in accuracy, and that this accuracy is considered inadequate for lunar missions. Second, a numerical method is presented for improving the accuracy to an acceptable level. Third, either the matrices from the strictly two-body computations or the improved ones can be used for midcourse guidance at the expense of additional corrective velocity. The magnitudes of the velocity penalties for the improved and unimproved matrices are compared. Fourth, a qualitative evaluation of the computer time and storage requirements for the approximate method is presented.

Theory

At this point, a brief outline of the method used for computing the transition matrices in the midcourse guidance studies mentioned earlier will help to illustrate the object of the present study. The computation was carried out using the components of position and velocity as variational parameters; however, the same approach could conceivably be used with other variational parameters. The procedure is as follows: A set of linear differential perturbation equations is obtained by expanding the vehicle's equations of motion about the reference trajectory. The coefficients in the equations are prescribed functions of position on the reference trajectory, hence, of time. The simultaneous integration of six sets of these equations with appropriate initial conditions will yield the desired matrices. The accuracy of this calculation is limited only by the accuracies of the mathematical model and of the numerical operations, but the integration is quite time consuming. In addition, if a high precision numerical integration is used, a large amount of computer storage may be required.

Use of Two-Body Approximation

One approach to the problem of simplifying the computation of the matrices is the use of two-body approximations. That is if the trajectory is assumed to be a conic over a given time interval, then the state vectors* at the initial and final times can be related to each other in closed form. Hence, the transition matrix over this interval can also be found in closed form. If the total trajectory is divided into n of these intervals, then the transition matrix relating perturbations from the reference trajectory at the final time to those at the initial time can be approximated by multiplying the n individual two-body matrices together.

As n is increased, that is, as the size of the intervals is reduced, the individual conics become better approximations to the actual trajectory. Therefore, if no accuracy is lost in the numerical operations, the matrices are obtained with increasingly greater accuracy.

There is, however, a limit to the accuracy which can be obtained by the reduction of the time intervals over which the reference trajectory is approximated by individual conics. To illustrate this point, consider the linear perturbation equations of motion

$$\dot{\bar{x}} = F\bar{x} \tag{1}$$

where $\bar{x}$ is the vector of perturbations from the reference trajectory and F is a matrix of coefficients which are functions of position on the reference trajectory. It can be shown that the transition matrix, Φ, also satisfies equation (1); that is

$$\dot{\Phi} = F\Phi \tag{2}$$

The matrix Φ is defined as the transition matrix which would be obtained if only a single homogeneous spherical central body were present and F_c as the value the matrix F would have in such a case. The matrices ψ and F_p can then be defined by the following equations:

$$\left. \begin{array}{c} \Phi = \varphi + \psi \\ F = F_c + F_p \end{array} \right\} \tag{3}$$

Equation (2) can now be written as

$$\dot{\varphi} + \dot{\psi} = F_c\varphi + F_p\varphi + F_c\psi + F_p\psi \tag{4}$$

but

$$\dot{\varphi} = F_c\varphi \tag{5}$$

so

$$\dot{\psi} = F_p\varphi + F_c\psi + F_p\psi \tag{6}$$

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*For the present, "state vector" will be defined as any set of six variables from which the vehicle's instantaneous position and velocity can be determined.

Now consider the case illustrated in the sketch below. The reference trajectory is being approximated by conics over a series of short time intervals for the purpose of computing the transition

CENTRAL BODY
REFERENCE TRAJECTORY ——
APPROXIMATING CONICS ---
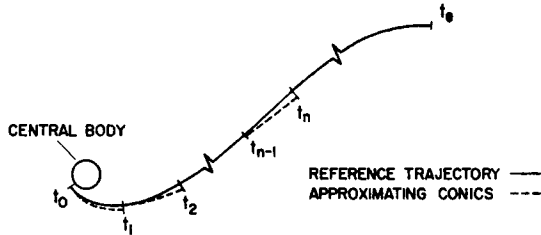
matrices. Here t_E is the terminal reference time and t_o is the time of injection. The transition matrix $\Phi(t_n; t_{n-1})$ from t_{n-1} to t_n is written, for convenience, as Φ_n and similarly for φ and ψ.

From equation (6)

$$\psi_n = \int_{t_{n-1}}^{t_n} (F_p \varphi_n + F_c \psi_n + F_p \psi_n) dt \qquad (7)$$

and ψ_n is the error in approximating the transition matrix, Φ_n, across the nth interval with the matrix φ_n computed from the two-body equations.

For use later in the paper, it is desired to know the limit approached by ψ_n as n becomes very large, that is, as the time interval approaches zero. The F matrices on any practical trajectory will be non-null and finite if the proper variables are chosen to describe the trajectory. Therefore, since at time t_{n-1}, φ_n is the unit matrix and ψ_n is the null matrix, the last two terms in the integrand become negligibly small with respect to the first as the time interval is made very small. Thus, according to the classic definition of the integral

$$\lim_{\Delta t \to o} \psi_n = F_p(t_n)\Delta t \qquad (8)$$

where $\Delta t = t_n - t_{n-1}$.

The transition matrix, Φ_E, relating deviations in the state vector at time t_E to those at t_o can be written in the following form:

$$\Phi_E = (\varphi_n + \psi_n)(\varphi_{n-1} + \psi_{n-1})\cdots(\varphi_2 + \psi_2)(\varphi_1 + \psi_1) \qquad (9)$$

In order to find the relationship between Φ_E and its two-body approximation, it is necessary to expand equation (9). Setting n equal to 4 is sufficient to show the form of the expansion. In this case

$$\begin{aligned}\Phi_E &= \varphi_4\varphi_3\varphi_2\varphi_1 + (\varphi_4 + \psi_4)(\varphi_3 + \psi_3)(\varphi_2 + \psi_2)\,\psi_1 \\ &\quad + (\varphi_4 + \psi_4)(\varphi_3 + \psi_3)\psi_2\varphi_1 + (\varphi_4 + \psi_4)\psi_3\varphi_2\varphi_1 + \psi_4\varphi_3\varphi_2\varphi_1 \\ &= \varphi_4\varphi_3\varphi_2\varphi_1 + \varphi_4\varphi_3\varphi_2\psi_1 + \varphi_4\varphi_3\psi_2\varphi_1 \\ &\quad + \varphi_4\psi_3\varphi_2\varphi_1 + \psi_4\varphi_3\varphi_2\varphi_1\end{aligned} \qquad (10)$$

Equation (10) can be generalized to n terms to give

$$\Phi(t_E;t_o) = \varphi(t_E;t_o) + \sum_{i=1}^{n} \Phi(t_E;t_i)\,\psi_i\varphi(t_{i-1};t_o) \qquad (11)$$

The second term on the right represents the error, $\psi(t_E;t_o)$, in approximating the transition matrices over the short intervals with two-body equations.

As n is allowed to become very large, equation (11) becomes

$$\Phi(t_E;t_o) = \varphi(t_E;t_o) + \lim_{n\to\infty} \sum_{i=1}^{n} \Phi(t_E;t_i)\,\psi_i\varphi(t_{i-1},t_o) \qquad (12)$$

but from equation (8)

$$\psi_i = F_p(t_i)\Delta t$$

so that

$$\Phi(t_E;t_o) = \varphi(t_E;t_o) + \lim_{n\to\infty} \sum_{i=1}^{n} \Phi(t_E;t_i)F_p(t_i)\Delta t \varphi(t_{i-1};t_o)$$

or

$$\Phi(t_E;t_o) = \varphi(t_E;t_o) + \int_{t_o}^{t_E} \Phi(t_E;t)F_p(t)\varphi(t;t_o)dt \qquad (13)$$

Thus in the limit the error, ψ_E, in approximating the transition matrix approaches the integral on the right hand side of equation (13). In order for this integral to yield the null matrix for all values of t_o the integrand must be identically the null matrix. Since the transition matrices are Jacobians of the variational parameters at one time, with respect to those at a previous time, they must be nonsingular. Some of the terms of the matrix F_p are the second partial derivatives of the perturbing potential function and, because of the form of that function, must be nonzero. For these reasons, the integrand in equation (13) cannot be identically zero and ψ_E must, in general, have nonzero terms. In other words, the integral in equation (13) represents a minimum error in approximating the transition matrices by closed-form computation.

It is of interest to know what effect the reduction of the time intervals has on the matrices from the closed-form computation. From equation (5) the closed-form matrix over the nth interval could be computed from

$$\varphi_n = \int_{t_{n-1}}^{t_n} F_c\varphi_n$$

where the terms of the matrix F_c are calculated as functions of the state vector on the approximating conic between t_{n-1} and t_n. As the time interval is made arbitrarily small, φ_n approaches its initial value, the unit matrix, and F_c approaches the value which would be calculated using the state vector on the reference trajectory at time t_n. From the classic definition of the integral

$$\lim_{n\to\infty} \varphi_n = I + F_c\Delta t \qquad (14)$$

where F_c is evaluated on the reference trajectory. If φ_E is written as the product to n matrices of the form in equation (14), then it can be expanded to give

$$\varphi_E = I + F_c(t_1)\Delta t + F_c(t_2)\varphi(t_1;t_0)\Delta t$$
$$+ F_c(t_3)\varphi(t_2;t_0)\Delta t + \ldots F_c(t_n)\varphi(t_{n-1};t_0)\Delta t$$

Hence,

$$\lim_{n\to\infty}\varphi_E = \int_{t_0}^{t_E} F_c(t)\varphi(t;t_0)dt \qquad (15)$$

where $\varphi(t_0) = I$ and F_c is evaluated on the reference trajectory. The integral in equation (15) represents the ultimate accuracy available, except in isolated fortuitous cases, from using closed-form equations to approximate the matrices. Therefore, this integral is useful as a standard of comparison for the closed-form matrices computed using finite time intervals.

Computation of Matrices in Closed Form

For convenience, the two-body matrices were computed in closed form with the components of the Cartesian position and velocity used both as the variational parameters* and as the components of the state vector. The procedure is one of linearization, as in the case of the differential equations, which uses a Taylor series expansion in which all derivatives of higher than first order are neglected.

The equations relating position and velocity at two points are written according to the method of Laplace[5] and differentiated. The closed-form equations are:

$$\left. \begin{array}{c} R = f\bar{R}_0 + g\bar{V}_0 \\ V = \dot{f}\bar{R}_0 + \dot{g}\bar{V}_0 \end{array} \right\} \qquad (16)$$

where $\bar{R}_0$ and $\bar{V}_0$ are the initial position and velocity vectors, $\bar{R}$ and $\bar{V}$ are their final values, while f, g, $\dot{f}$, and $\dot{g}$ are scalar functions of the initial position and velocity and of the time interval being used.

The first partial derivatives of the above equation are:

$$\left. \begin{array}{l} \dfrac{\partial\bar{R}}{\partial\bar{R}_0} = fI + \bar{R}_0(\nabla_R f)^T + \bar{V}_0(\nabla_R g)^T \\[2mm] \dfrac{\partial\bar{R}}{\partial\bar{V}_0} = gI + \bar{R}_0(\nabla_V f)^T + \bar{V}_0(\nabla_V g)^T \\[2mm] \dfrac{\partial\bar{V}}{\partial\bar{R}_0} = \dot{f}I + \bar{R}_0(\nabla_R \dot{f})^T + \bar{V}_0(\nabla_R \dot{g})^T \\[2mm] \dfrac{\partial\bar{V}}{\partial\bar{V}_0} = \dot{g}I + \bar{R}_0(\nabla_V \dot{f})^T + \bar{V}_0(\nabla_V \dot{g})^T \end{array} \right\} \qquad (17)$$

*It is well to point out that Pines[3,4] has shown that transition matrices computed over long time arcs become ill-conditioned as a result of secular terms and that this difficulty can be reduced greatly if variational parameters different from the ones used here are chosen. No difficulty because of this phenomenon has been encountered in the Ames lunar studies, presumably because the time arcs involved have never exceeded about half an orbital period. Such ill-conditioning need not affect the comparison of results from an approximate method and a more accurate model, but it should be considered if the method is to be incorporated into a navigation system.

The vector partial derivatives in equation (17) are of the form

$$\frac{\partial\bar{R}}{\partial\bar{R}_0} = \begin{bmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\[2mm] \dfrac{\partial y}{\partial x_0} & \cdot & \cdot & \cdot & \cdot & \cdot \\[2mm] \dfrac{\partial y}{\partial z_0} & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The notation ∇_R and ∇_V indicates the gradient with the components of position and velocity, respectively, used as independent variables, and I is the 3×3 unit matrix. The vector partial derivatives are submatrices of the two-body transition matrix which can be written:

$$\varphi = \begin{bmatrix} \dfrac{\partial\bar{R}}{\partial\bar{R}_0} & \dfrac{\partial\bar{R}}{\partial\bar{V}_0} \\[3mm] \dfrac{\partial\bar{V}}{\partial\bar{R}_0} & \dfrac{\partial\bar{V}}{\partial\bar{V}_0} \end{bmatrix} \qquad (18)$$

Improvement of Two-Body Matrices

It will be shown later that with finite time intervals the error in the matrices was considered unacceptable even though they compared favorably with the results of equation (15). This inaccuracy was the reason for performing the second part of the work, that is, finding a means for improving the approximation. The approach to this problem was to perform an approximate integration of equation (7); that is, of

$$\psi_n = \int_{t_{n-1}}^{t_n} (F_p\varphi_n + F_c\psi_n + F_p\psi_n)dt$$

If Cartesian coordinates are used, the matrices in equation (7) can be partitioned in the following form:

$$\left. \begin{array}{c} F_c = \begin{bmatrix} 0 & I \\ \bar{F}_c & 0 \end{bmatrix} \\[4mm] F_p = \begin{bmatrix} 0 & 0 \\ \bar{F}_p & 0 \end{bmatrix} \\[4mm] \varphi = \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\bar{R}}{\partial\bar{R}_0} & \dfrac{\partial\bar{R}}{\partial\bar{V}_0} \\[3mm] \dfrac{\partial\bar{V}}{\partial\bar{R}_0} & \dfrac{\partial\bar{V}}{\partial\bar{V}_0} \end{bmatrix} \end{array} \right\} \qquad (19)$$

In this notation the zeros are 3×3 null matrices, and $\bar{F}_c$ and $\bar{F}_p$ are 3×3 matrices of partial derivatives of accelerations due to the spherical central body and of the perturbing accelerations, respectively. The matrix ψ is partitioned in a form corresponding to that of φ.

Substituting equations (19) in equation (6) gives

$$\begin{bmatrix} \dot{\psi}_1 & \dot{\psi}_2 \\ \dot{\psi}_3 & \dot{\psi}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \overline{F}_p & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{bmatrix} + \begin{bmatrix} 0 & I \\ \overline{F}_c & 0 \end{bmatrix} \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ \overline{F}_p & 0 \end{bmatrix} \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{bmatrix} \qquad (20)$$

Expanding equation (20) results in

$$\left.\begin{array}{l} \dot{\psi}_1 = \psi_3 \\[4pt] \dot{\psi}_2 = \psi_4 \\[4pt] \dot{\psi}_3 = \overline{F}_p \varphi_1 + \overline{F}_c \psi_1 + \overline{F}_p \psi_1 \\[4pt] \dot{\psi}_4 = \overline{F}_p \varphi_2 + \overline{F}_c \psi_2 + \overline{F}_p \psi_2 \end{array}\right\} \qquad (21)$$

so that

$$\left.\begin{array}{l} \ddot{\psi}_1 = \overline{F}_p \varphi_1 + \overline{F}_c \psi_1 + \overline{F}_c \psi_1 \\[4pt] \ddot{\psi}_2 = \overline{F}_p \varphi_2 + \overline{F}_c \psi_2 + \overline{F}_p \psi_2 \end{array}\right\} \qquad (22)$$

If ψ_1 and ψ_2 are neglected,

$$\left.\begin{array}{l} \ddot{\psi}_1 \approx F_p \varphi_1 \\[4pt] \ddot{\psi}_2 \approx F_p \varphi_2 \end{array}\right\} \qquad (23)$$

It is assumed that the accelerations in equation (23) are constant at their average value over the nth interval. Since φ_1 at the beginning of the interval is the unit matrix and φ_2 is the null matrix

$$\psi_1(t_n;t_{n-1}) = \overline{F}_p(t_{n-1}) + \overline{F}_p(t_n)\varphi_1(t_n)\frac{(t_n - t_{n-1})^2}{4}$$

$$\psi_2(t_n;t_{n-1}) = \overline{F}_p(t_n)\varphi_2(t_n)\frac{(t_n - t_{n-1})^2}{4}$$

$$\psi_3(t_n;t_{n-1}) = \overline{F}_p(t_{n-1}) + F_p(t_n)\varphi_1(t_n)\frac{(t_n - t_{n-1})}{2}$$

$$\psi_4(t_n;t_{n-1}) = \overline{F}_p(t_n)\varphi_2(t_n)\frac{(t_n - t_{n-1})}{2} \qquad (24)$$

Equations (24) represent, in partitioned form, the results of an approximate integration of equation (7). The perturbation matrices thus computed can be added to the two-body matrices for corresponding intervals in order to improve the approximation.

Evaluation of Accuracy

The significance of the inherent inaccuracy in the closed-form matrices and the effectiveness of the improvement have been judged on the basis of a numerical example. Likewise, a comparison of the velocity penalties discussed earlier, resulting from use of the improved and unimproved matrices, was carried out for the same numerical example. This section of the paper describes the method of evaluating the errors in prediction and guidance, the error criteria used, and the method of computing the velocity penalty.

Method of evaluation.- The analysis of the accuracy of the approximate method is carried out by comparison of the approximate matrices with those of a "correct" solution. The correct solution in this study is assumed to be obtained by the integration of the perturbation differential equations from a mathematical model considered sufficiently accurate for computing the reference trajectory. For determination of the errors in prediction, an error matrix, E, is found by subtracting the approximate matrix from the correct one. This error matrix is partitioned in the form of equation (18).

$$E = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} = \begin{bmatrix} \Delta\frac{\partial \overline{R}}{\partial \overline{R}_0} & \Delta\frac{\partial \overline{R}}{\partial \overline{V}_0} \\[8pt] \Delta\frac{\partial \overline{V}}{\partial \overline{V}_0} & \Delta\frac{\partial \overline{V}}{\partial \overline{V}_0} \end{bmatrix} \qquad (25)$$

The matrix on the extreme right indicates which of the E_i corresponds to the error in partial derivatives of position with respect to initial position, position with respect to initial velocity, etc. For example, if an initial position deviation from the reference were multiplied by E_1 the result would be the vector error in predicting position deviation from the reference at the final time. The E_i will be referred to in the remainder of the paper as the prediction error matrices.

It is also desired to know the error arising from using the approximate matrices for guidance. For this purpose the fixed time of arrival guidance law from previous Ames studies[1,2] will be used. The transition matrix Φ from the time of a velocity correction to the terminal point is partitioned in the form described previously

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix}$$

Then the velocity vector, $\delta\overline{V}_G$, to be gained is

$$\delta\overline{V}_G = -\Phi_2^{-1}\Phi_1\delta\overline{R} - \delta\overline{V} \qquad (26)$$

where $\delta\overline{R}$ and $\delta\overline{V}$ are the position and velocity deviations from the reference trajectory at the time of the correction. If Φ, $\delta\overline{R}$, and $\delta\overline{V}$ are known exactly, the position deviation, $\delta\overline{R}_E$, at the end point after the application of the velocity correction will be

$$\delta\overline{R}_E = \Phi_1\delta\overline{R} + \Phi_2\delta\overline{V} - \Phi_2(\Phi_2^{-1}\Phi_1\delta\overline{R} + \delta\overline{V}) = 0 \qquad (27)$$

Using the approximate matrices, φ_1 and φ_2, in computing the velocity correction gives

$$\delta\overline{R}_E = (\Phi_1 - \Phi_2\varphi_2^{-1}\varphi_1)\delta\overline{R} = B\delta\overline{R} \qquad (28)$$

The matrix B will be referred to as the guidance error matrix.

The results to be presented later show that as the terminal point is approached, the guidance errors resulting from use of the approximate matrices become negligibly small. Therefore, a guidance error resulting from an earlier correction can be eliminated at the expense of a subsequent correction nearer the terminal point. It is assumed that this second correction will be made at some time, t_2, when the residual error will be negligible. The

deviations from the reference at this time will be given by

$$\begin{bmatrix} \delta \overline{R}_2 \\ \delta \overline{V}_2 \end{bmatrix} = \Phi^{-1}(t_E;t_2) \begin{bmatrix} \delta \overline{R}_E \\ \delta \overline{V}_E \end{bmatrix} \qquad (29)$$

The terminal position deviation $\delta \overline{R}_E$ is given by equation (28). Similarly, the velocity deviation, $\delta \overline{V}_E$, can be shown to be

$$\delta \overline{V}_E = (\Phi_3 - \Phi_4 \varphi_2^{-1} \varphi_1) \delta \overline{R} = C \delta \overline{R} \qquad (30)$$

where $\delta \overline{R}$ and the matrix C are evaluated at the time of the first correction.

It is shown in reference 2 that if Φ is partitioned as before, then

$$\Phi^{-1} = \begin{vmatrix} \Phi_4^T & -\Phi_2^T \\ -\Phi_3^T & \Phi_1^T \end{vmatrix} \qquad (31)$$

Therefore, with $\delta \overline{R}_E$ and $\delta \overline{V}_E$ substituted from equations (28) and (30) and the subscript 1 added to indicate the time of the first velocity correction,

$$\begin{bmatrix} \delta \overline{R}_2 \\ \delta \overline{V}_2 \end{bmatrix} = \begin{bmatrix} \Phi_4^T & -\Phi_2^T \\ -\Phi_3^T & \Phi_1^T \end{bmatrix} \begin{bmatrix} B_1 \delta \overline{R}_1 \\ C_1 \delta \overline{R}_1 \end{bmatrix} \qquad (32)$$

The second velocity correction is given by

$$\delta \overline{V}_{G_2} = -\varphi_2^{-1} \varphi_1 \delta \overline{R}_2 - \delta \overline{V}_2 \qquad (33)$$

where it is understood that the Φ_i and φ_i in equations (32) and (33) are evaluated at the time of the second correction. Values of $\delta \overline{R}_2$ and $\delta \overline{V}_2$ for substitution into equation (33) are obtained from equation (32).

$$\delta \overline{V}_{G_2} = (\Phi_3^T - \varphi_2^{-1} \varphi_1 \Phi_4^T) B_1 \delta \overline{R}_1 + (\varphi_2^{-1} \varphi_1 \Phi_2^T - \Phi_1^T) C_1 \delta \overline{R}_1 \qquad (34)$$

If it is assumed that the second correction is made at a range where the differences between the correct and approximate matrices are negligible, then equation (34) becomes

$$\delta \overline{V}_{G_2} \approx (\Phi_3^T - \Phi_2^{-1} \Phi_1 \Phi_4^T) B_1 \delta \overline{R}_1 + (\Phi_2^{-1} \Phi_1 \Phi_2^T - \Phi_1^T) C_1 \delta \overline{R}_1 \qquad (35)$$

From equation (31) it can be shown that

$$\Phi_2^{-1} \Phi_1 = \Phi_1^T (\Phi_2^{-1})^T$$

$$\Phi_1 \Phi_4^T = I + \Phi_2 \Phi_3^T$$

so that

$$\delta \overline{V}_{G_2} \approx -\Phi_2^{-1} B_1 \delta \overline{R}_1 = D \delta \overline{R}_1 \qquad (36)$$

The matrix D will be referred to as the velocity penalty matrix.

It is difficult to assess the velocity penalty and the errors in prediction and guidance by considering the individual terms in the matrix, and a simpler criterion is needed. One possibility would be to multiply each matrix by its transpose and compute

the eigenvalues of the resulting matrix. The square root of the maximum eigenvalue in each case would then represent the maximum velocity penalty, prediction error, or guidance error resulting from using the approximate matrices with a unit deviation from the reference trajectory. Instead of this maximum value, the error was represented by the norm of the appropriate matrix because of the computational simplicity. Since the norm is equal to the square root of the sum of the eigenvalues, the resulting information is essentially the same.

An idea of the error in prediction relative to the total predicted error can be obtained by expressing the norm of the error matrices as percentages of the norms of the corresponding correct submatrices. The guidance error, due to use of the approximate matrices, is a function only of the initial position deviation. Therefore, a similar percentage error in guidance can be found from the norm of the guidance error matrix expressed as a percentage of the norm of the correct submatrix of partials of position with respect to position. The velocity penalty is also a function only of the initial position deviation, so that computation of a relative velocity correction penalty should involve only that portion of the initial correction arising from position deviations from the reference. The criterion used for this comparison is the ratio of the norm of the velocity penalty matrix to that of the matrix $\varphi_2^{-1} \varphi_1$, computed at the time of the initial correction.

Digital computer study.- The accuracy was evaluated by means of a digital computer study of a sample moon-to-earth trajectory. The computational methods are described here briefly. The trajectory has a flight time of about 3-1/4 days from perilune to return perigee and lies approximately in the moon's orbital plane. The flight time is fairly long and therefore allows the perturbing forces not accounted for in the two-body equations to have a relatively large effect. For comparison purposes, the transition matrices obtained from integrating the differential perturbation equations from a four-body model are taken as correct. This model includes the second harmonic term of the earth's gravitational potential as well as the vehicle, sun, earth, and moon.

The two-body transition matrices were computed, as described in the section on theory, both by integration and in closed form. It was found better in this computation to take a larger sphere of influence for the moon than the generally accepted value given by the 2/5 power of the mass ratios. The reason for this can be seen if the perturbation differential equations instead of the vehicle's equations of motion are used to compute the moon's sphere of influence. As a result of differentiation, the term representing the action of the central body on the perturbing body disappears and the perturbing actions are equal where the accelerations due to the two bodies are equal. In the one-dimensional case, this equality occurs when the ratio of the ranges to the two bodies is equal to the cube root of the mass ratios.

The integration for both the reference trajectory and the perturbation equations uses a Cowell "second-sum" method. A fourth-order Runge-Kutta method is used to start the integration and to change step size during the flight. The equations given in the section on theory were used to compute the conic approximations, and the time intervals over which the conic approximations were used were

chosen to coincide with an integral number of intervals from the Cowell integration.

Results and Discussion

The data resulting from the digital computer study will now be presented. These results will, first, indicate why the unimproved two-body approximation is considered inadequate; second, show that the suggested improvement is adequate; third, compare the velocity penalties resulting from the use of improved and unimproved matrices.

Accuracy of Unimproved Matrices

The norms of the prediction error matrices resulting from integrating the two-body perturbation differential equations along the reference trajectory are presented in figure 1. The prediction error matrices from equation (25) are indicated for the appropriate curves. The prediction submatrix to which the error matrix corresponds is also indicated along with the units used. Note that the increase in prediction error, with range from the earth, is approximately exponential. The norms of the corresponding submatrices of the correct transition matrices increase in a similar fashion with increasing range. For this reason, the percentage errors in prediction exhibit a much less drastic increase.

These percentages for the data just presented are plotted in figure 2. The curves for prediction error matrices E_3 and E_4 have been omitted because they are nearly identical with those for matrices E_1 and E_2, respectively. Note that even though the error in prediction increases continually with range from the earth, the percentage error decreases near the moon. This decrease occurs, of course, because the norms of the correct prediction submatrices increase more rapidly in this region than those of the error matrices.

For the closed-form computation of the two-body matrices the time intervals used were sufficiently short to give results close to those given by the integration. A single time interval was satisfactory for ranges less than about 330,000 km, but for the portion of the trajectory between that range and perilune, it was found necessary to rectify more often. Intervals of about 0.1 day were found to give satisfactory results and were used over the entire trajectory.

The percentage errors in prediction for the closed-form computation are presented in figure 3. Comparison of these curves with those of figure 2 shows little difference in the accuracy of the two methods of computation except in the region of greatest error. The maximum percentage for E_1 from the closed-form computation is about 13.5 percent as compared to 12.5 percent for the integration.

The norms of the guidance error matrices for the two methods of computation are plotted together in figure 4. The correspondence between the two methods is quite good at ranges beyond 200,000 km. The pronounced differences between the two curves at the shorter ranges occur because the earth's oblateness causes larger relative differences between the reference trajectory and the conic approximations. The errors for both methods are quite small in this region and the difference between the two is not significant when the guidance error is considered on a percentage basis as in figure 5.

As in the case of the prediction error, the only noticeable difference between the percentage errors for the two methods of computation occurs near the range of 350,000 km. The data in figure 5 indicate that the two-body matrices could probably be used successfully for guidance at ranges of less than 250,000 km. The large errors at longer ranges indicate the need for a correction method such as was outlined earlier in the paper. The next data to be presented will show the improvement in accuracy resulting from use of the correction method.

Accuracy of the Improved Matrices

The same 0.1 day time intervals used for the closed-form computation just discussed were used to compute the corrected matrices. It was found that including the earth's oblateness terms in the correction computation decreases rather than improves the accuracy unless the time intervals are greatly reduced near the earth. For this reason the oblateness terms were omitted from this calculation, and the resulting percentage errors in prediction are presented in figure 6. In this case all the errors are less than 0.5 percent. The irregular form of the curves is to be expected because of the assumptions made in computing the corrections. These fluctuations, as well as the average value, could be reduced to the limits of computational accuracy by reduction of the time intervals and inclusion of the earth's oblateness.

The improvement in guidance accuracy resulting from the use of the corrected matrices is illustrated in figure 7. For comparison, the norms of the guidance error matrices from the corrected and uncorrected closed-form computations are both presented. Except at short ranges, where oblateness is a significant factor, the guidance error is reduced by at least an order of magnitude at every point. The percentage error in guidance was also calculated using the corrected matrices and was found to reach a maximum value of about 0.2 percent at a range of 150,000 km.

The Velocity Penalty

The final data to be presented compare the velocity penalties resulting from use of the improved and unimproved matrices. For the purpose of computing the velocity penalty that resulted from using the approximate matrices, it was assumed that the final velocity correction would be made at the shortest range for which data have been presented so far, about 45,000 km. It was further assumed, as discussed in the theory, that at this range the errors in the approximate matrices are negligible. The norms of the velocity penalty matrices computed at the longer ranges from the closed-form transition matrices, both corrected and uncorrected, are presented in figure 8. The additional velocity correction required is quite small in both cases for corrections made at ranges of less than 200,000 km. At longer ranges, however, the penalty for using the uncorrected matrices rises rapidly, while that for the corrected matrices remains quite small.

The ratio of the norm of the velocity penalty matrix to that associated with the initial correction is plotted in figure 9 for both cases discussed above. Elimination of the errors arising from the use of the approximate matrices at long ranges requires a velocity penalty of as much as seven times the initial correction. Of course, the use of additional corrections at intermediate ranges and

the proper choice of a correction schedule, would reduce the penalty from that shown here. On the other hand, it seems apparent from these results that the use of the uncorrected two-body matrices will result in a substantial velocity penalty. The corrected matrices result in a maximum velocity penalty of about 6 percent of the initial correction and this penalty also may be reduced by proper scheduling.

Qualitative Evaluation of Computer Requirements

Finally, it is desired to consider whether the use of the approximate method has any advantages in terms of required computer time and storage. The study presented here was carried out on an IBM 7090 computer and the IBM Fortran compiler. For this reason, any precise comparison of computer storage requirements is impractical, but some general conclusions can be drawn.

First, if a Cowell integration is used for the reference trajectory, it has been found in another Ames study that the integration of the perturbation equations using a fourth-order Runge-Kutta method requires less storage than is needed for the closed-form computation of the two-body matrices. No study has been conducted on the accuracy of this integration, but a few preliminary results indicate about the same accuracies as in the corrected matrices described here. This integration requires quite short step sizes, and for this reason it consumes about four times as much computer time as the approximate method. The need for such short steps indicates that it would be impractical to use this simple Cowell integration of the perturbation equations in conjunction with an Encke integration of the reference trajectory.

In the case of an Encke integration of the reference trajectory, it would be necessary to compute the two-body matrices so that only the additional computer capacity required to account for the perturbing accelerations needs to be considered. The Encke differential equations could be expanded in a Taylor series about the reference trajectory in order to get the perturbation differential equations. These equations could then be integrated to give the corrections to the two-body transition matrices. The correction scheme discussed previously represents a simplification of this procedure, but a comparison of the computer requirements could be arrived at only by programming the two methods. Use of the correction method as described here will require approximately double the storage needed for computing only the two-body matrices.

Conclusions

The conclusions which have been drawn from the results of the study are listed below:

1. The transition matrices computed in closed form using conic approximations to the reference trajectory can be used for midcourse guidance at the expense of a substantial penalty in corrective velocity. While this velocity penalty is large compared with the basic midcourse requirements, it might be acceptable in terms of the over-all requirements of the mission.

2. The accuracy of the two-body matrices can be improved to almost any desired level at the expense of increased computer capacity and time.

The computation of these corrections approximately doubles the computer capacity required over that needed for computing the two-body matrices only.

3. If a Cowell integration is used for the reference trajectory, the approximate method appears to be inferior on the basis of computer storage requirements. The approximate method does show a substantial savings in time.

4. Since an Encke integration of the reference trajectory would require computation of the two-body matrices, only the additional computer capacity required for improving these matrices needs to be considered. The approximate method presented for improving the matrices is actually an approximate method of integrating the perturbed Encke differential equations, and therefore represents some saving in computer capacity and time. Considerably more study would be required to evaluate this savings.

References

1. Smith, Gerald L., Schmidt, Stanley F., and McGee, Leonard A.: Application of Statistical Filter Theory to the Optimal Estimation of Position and Velocity on Board a Circumlunar Vehicle. NASA TR R-135, 1962.

2. McLean, John D., Schmidt, Stanley F., and McGee, Leonard A.: Optimal Filtering and Linear Prediction Applied to a Midcourse Navigation System for the Circumlunar Mission. NASA TN D-1208, 1962.

3. Pines, Samuel, Wolf, Henry, and Payne, Mary: Asymptotically Singular Differential Correction Matrices. Rep. RAC 708-453 (328), Republic Aviation Corp., April 1961.

4. Pines, Samuel, and Sistino, Alphonse: The Use of Variational Parameters in Multivariable Linear Prediction and Filtering Theory. Rep. RAC 907 (ARD-947-450), Republic Aviation Corp., Sept. 1962.

5. Herget, Paul: The Computation of Orbits. Published privately by the author, 1948.
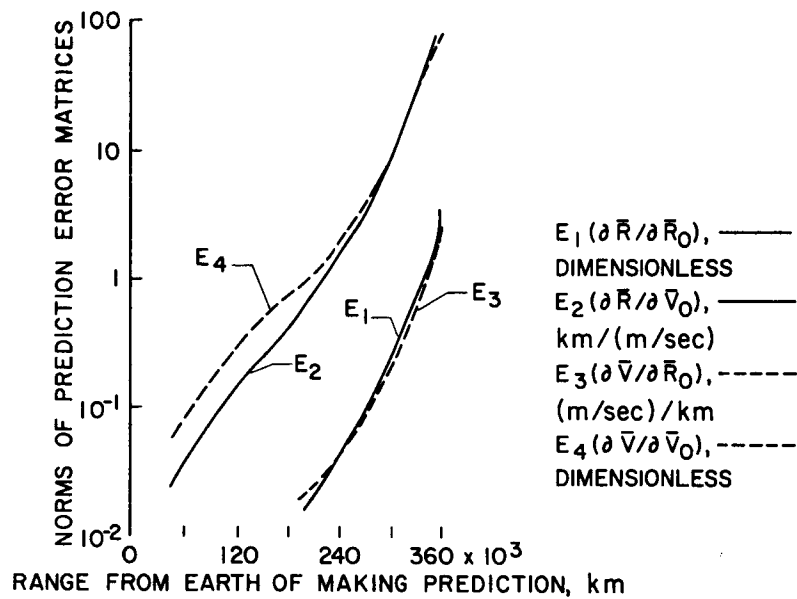
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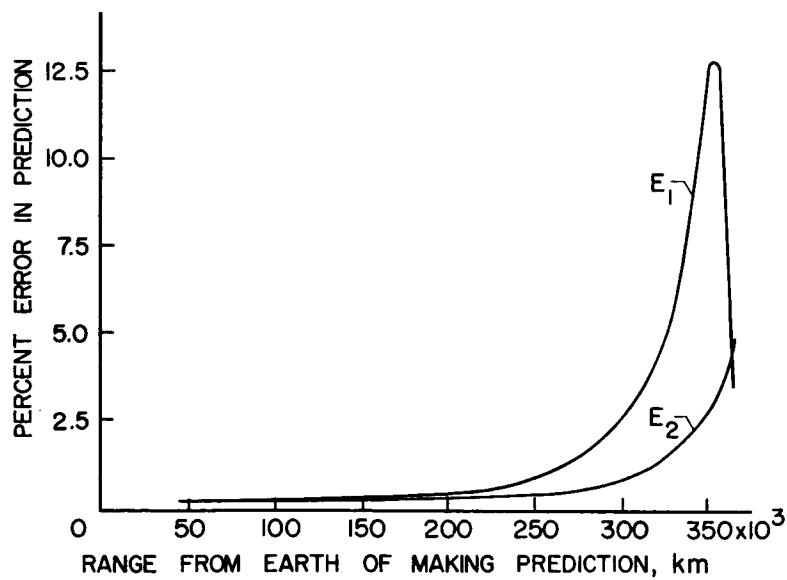Figure 1. - Errors from integration method.

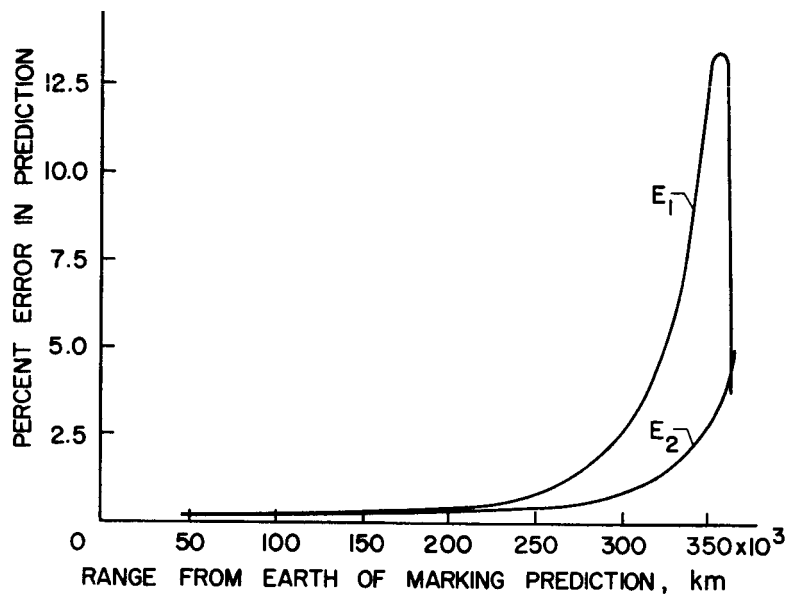Figure 2. - Percent prediction errors from integration.

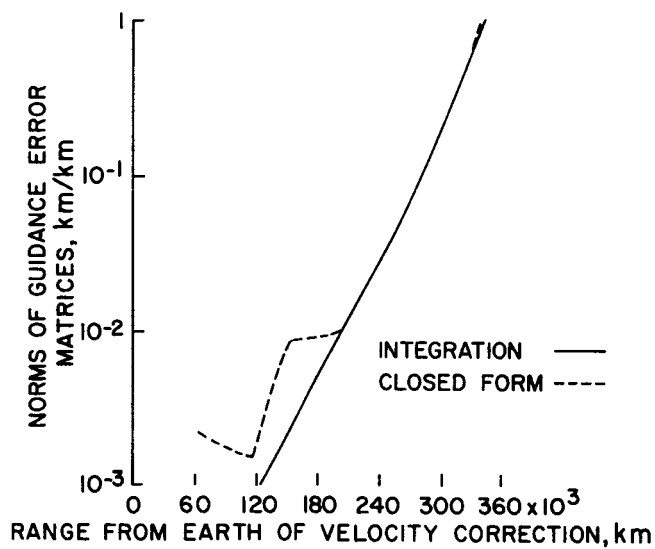Figure 3. - Percent prediction errors from closed form.
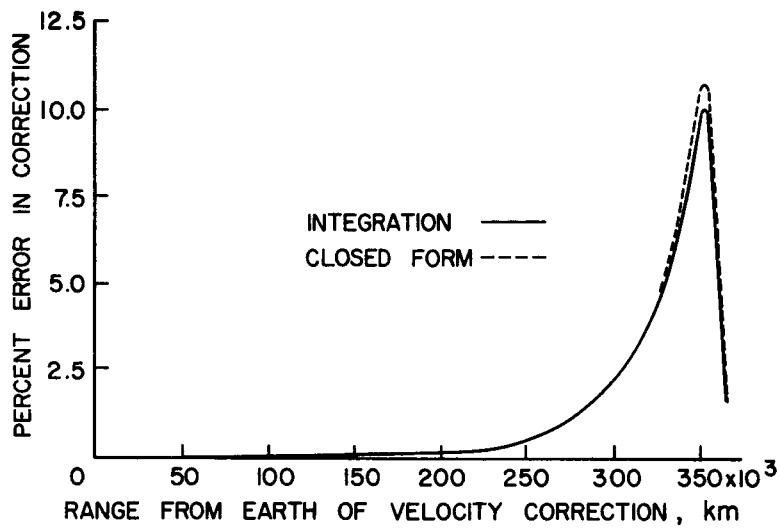
Figure 4. - Guidance errors from two-body matrices.

Figure 5. - Percent guidance errors from two-body matrices.
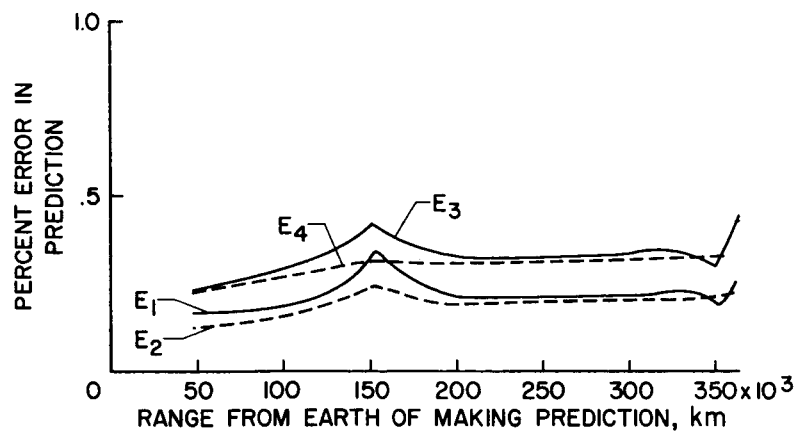
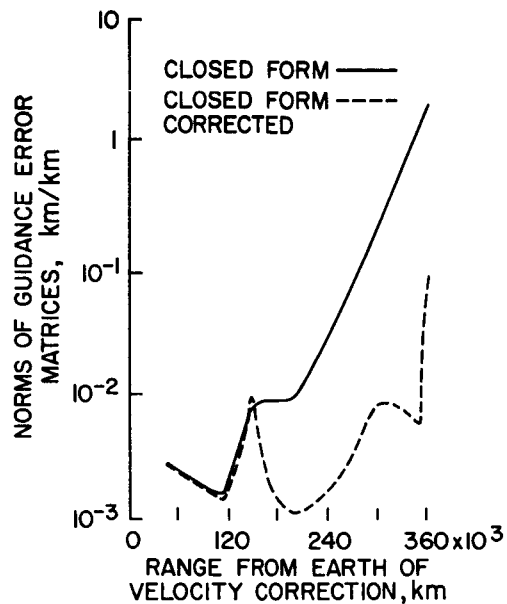Figure 6. - Percent prediction errors from corrected matrices.
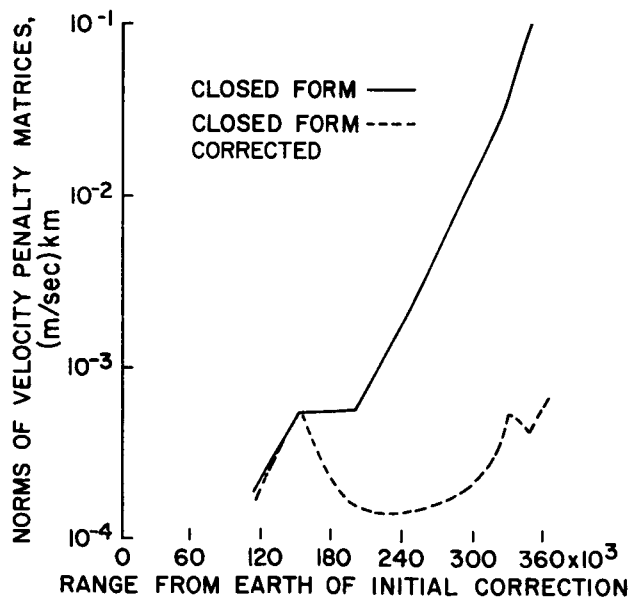
Figure 7. - Comparison of guidance errors.

Figure 8. - Velocity penalty.

Figure 9. - Relative size of velocity penalty.

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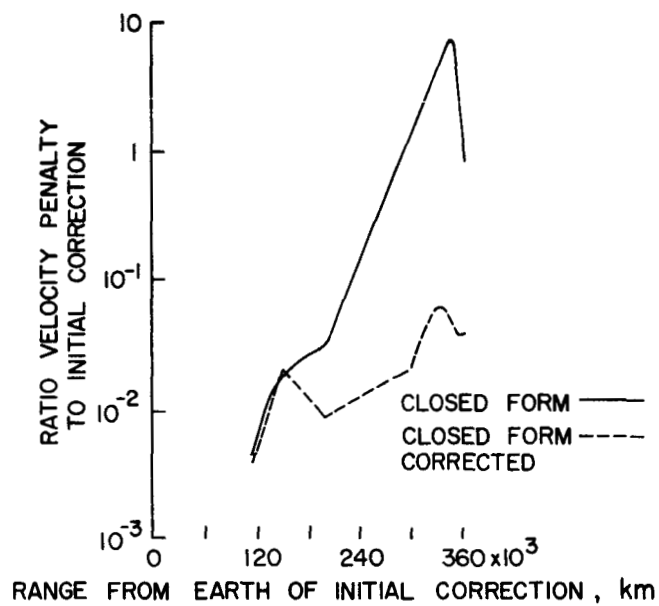
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